# ON SOME LOW-FREQUENCY VIBRATIONS OF A LIQUID LAYER IN AN ELASTIC MEDIUM

## (O NEKOTORYKH NIZKOCHASTOTNYKH KOLEBANIIAKH ZHIDKOGO SLOIA V UPRUGOI SREDE)

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P.V. KRAUKLIS
(Leningrad)

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It is well known that a liquid layer, situated between elastic half-spaces, in which the velocity of transverse waves is larger than the velocity of sound in the layer, is a wave guide [1]. In this case, the wave guide character of the propagation has been traced to interference of the waves in the layer with angles larger than the limiting one. More detailed investigation of the processes occurring in systems of the present type show that, when there is an arbitrary relation between the parameters of the media, there exists yet another type of vibration diminishing slowly with distance, and that the low-frequency part of this vibration has a slow velocity of propagation [2]. Some of the basic features of these vibrations will be studied below.

**1.** In a system of cylindrical coordinates $(r, \theta, z)$, let the liquid layer *1*, determined by the condition $0 \leqslant z \leqslant h$ separate the elastic half-space *0*, for which $z < 0$, from the half-space *2*, for which $z > h$. Each of these media is characterized by the longitudinal and transverse velocities of propagation $a_i^{-1}$ $(i = 0, 1, 2)$ and $b_i^{-1}$ $(i = 0, 2)$, respectively, and by the densities $\rho_i$ $(i = 0, 1, 2)$, which are connected with Lamé's constants

$$a_i^2 = \frac{\rho_i}{\lambda_i + 2\mu_i}, \qquad b_i^2 = \frac{\rho_i}{\mu_i} \tag{1.1}$$

At a point of the medium with coordinates $(0, 0, -H)$ there is a source like a center of dilation, the time dependence of which is expressed by a unit function. The source produces in the medium *0* a field of displacements, characterized by the potential

$$\varphi_0^\circ = \int_0^\infty \frac{I_0(kr)}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} X_0^\circ(k, \eta) \exp\left[k\left(\frac{t}{b_0}\eta \pm (z + H)\alpha_0\right)\right] d\eta\, dk \tag{1.2}$$

where

$$X_0^\circ = \frac{1}{4\pi(\lambda_0 + 2\mu_0)\,\eta\,\alpha_0}, \qquad \alpha_0 = \sqrt{1 + \gamma_0^2\eta^2}, \qquad \gamma_0 = \frac{a_0}{b_0} \tag{1.3}$$

In this a cut is to be made in the $\eta$-plane between the points $\pm i/\gamma_0$ in the left-hand half-plane, and the branch of the square root is fixed by the condition that arg $\alpha_0 = 0$ when $\eta > 0$. In formula (1.2), the positive sign is to be taken when $z < -H$ and the negative sign when $z > -H$.

We denote by $\varphi_0$, $\psi_0$, $\varphi_2$, $\psi_2$ the longitudinal and transverse potentials of the auxiliary fields of elastic displacements in the half-spaces *0* and *2*, and by $\varphi_1$ the potential of the displacement field in the liquid layer *1*. The potentials must satisfy the equations

$$\frac{\partial^2\varphi_\nu}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_\nu}{\partial r} + \frac{\partial^2\varphi_\nu}{\partial z^2} = a_\nu^2\frac{\partial^2\varphi_\nu}{\partial t^2} \qquad (\nu = 0, 2) \tag{1.4}$$

$$\frac{\partial^2\psi_\nu}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_\nu}{\partial r} + \frac{\partial^2\psi_\nu}{\partial z^2} - \frac{\psi_\nu}{r^2} = b_\nu^2\frac{\partial^2\psi_\nu}{\partial t^2}, \qquad \frac{\partial^2\varphi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_1}{\partial r} + \frac{\partial^2\varphi_1}{\partial z^2} = a_1^2\frac{\partial^2\varphi_1}{\partial t^2}$$

the zero initial conditions

$$\varphi_1 = \varphi_2 = \varphi_0 = \psi_0 = \psi_2 = 0, \qquad \frac{\partial\varphi_1}{\partial t} = \frac{\partial\varphi_2}{\partial t} = \frac{\partial\varphi_0}{\partial t} = \frac{\partial\psi_0}{\partial t} = \frac{\partial\psi_2}{\partial t} = 0 \qquad (t \leqslant 0) \tag{1.5}$$

and the boundary conditions

$$\begin{gathered}
\frac{\partial\varphi_0}{\partial z} + \frac{\partial\psi_0}{\partial r} + \frac{\psi_0}{r} = \frac{\partial\varphi_1}{\partial z} \\
\mu_0\left[(b_0^2 - 2a_0^2)\frac{\partial^2\varphi_0}{\partial t^2} + 2\frac{\partial^2\varphi_0}{\partial z^2} + 2\frac{\partial^2\psi_0}{\partial r\,\partial z} + \frac{2}{r}\frac{\partial\psi_0}{\partial z}\right] = \rho_1\frac{\partial^2\varphi_1}{\partial t^2} \quad \text{when } z = 0 \\
\mu_0\left[2\frac{\partial^2\varphi_0}{\partial r\,\partial z} + b_0^2\frac{\partial^2\psi_0}{\partial t^2} - 2\frac{\partial^2\psi_0}{\partial z^2}\right] = 0 \\
\frac{\partial\varphi_2}{\partial z} + \frac{\partial\psi_2}{\partial r} + \frac{\psi_2}{r} = \frac{\partial\varphi_1}{\partial z} \\
\mu_2\left[(b_2^2 - 2a_2^2)\frac{\partial^2\varphi_2}{\partial t^2} + \frac{\partial^2\varphi_2}{\partial z^2} + 2\frac{\partial^2\psi_2}{\partial r\,\partial z} + \frac{2}{r}\frac{\partial\psi_2}{\partial z}\right] = \rho_1\frac{\partial^2\varphi_1}{\partial t^2} \quad \text{when } z = h \\
\mu_2\left[2\frac{\partial^2\varphi_2}{\partial r\,\partial z} + b_2^2\frac{\partial^2\psi_2}{\partial t^2} - 2\frac{\partial^2\psi_2}{\partial z^2}\right] = 0
\end{gathered} \tag{1.6}$$

which insure the continuity of the vertical components of the displacement vectors and of the vertical components of stress $t_{zz}$ and the vanishing of shear stresses.

The solution of the indicated problem can without difficulty be represented in the form

$$\varphi_0 = \int_0^\infty \frac{I_0(kr)\,dk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} X_0\exp\left[k\left(\frac{t}{b_0}\eta + z\alpha_0\right)\right]d\eta$$

$$\psi_0 = \int_0^\infty \frac{I_1(kr)\,dk}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} Y_0 \exp\left[k\left(\frac{t}{b_0}\eta + z\beta_0\right)\right] d\eta$$

$$\varphi_1 = \int_0^\infty \frac{I_0(kr)\,dk}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} [X_1^- \sinh kz\alpha_1 + X_1^+ \cosh kz\alpha_1] \exp\frac{kt\eta}{b_0}\, d\eta \qquad (1.7)$$

$$\varphi_2 = \int_0^\infty \frac{I_0(kr)\,dk}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} X_2 \exp\left[k\left(\frac{t}{b_0}\eta - (z-h)\,\alpha_2\right)\right] d\eta$$

$$\psi_2 = \int_0^\infty \frac{I_1(kr)\,dk}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} Y_2 \exp\left[k\left(\frac{t}{b_0}\eta - (z-h)\,\beta_2\right)\right] d\eta$$

where

$$\alpha_\nu = \sqrt{1+\gamma_\nu^2\eta^2} \quad (\nu = 0, 1, 2), \qquad \beta_0 = \sqrt{1+\eta^2}, \qquad \beta_2 = \sqrt{1+\delta_2^2\eta^2}$$

$$\gamma_i = \frac{a_i}{b_0}, \qquad \delta_2 = \frac{b_2}{b_0}$$

The above square roots can be determined by the conditions that $\arg \alpha_i = 0$, $\arg \beta_i = 0$ when $\eta > 0$, and the cuts from the branch points of the square roots can be drawn in the left-hand half-plane parallel to the real axis. The functions $X_i$ and $Y_i$ can be determined from the boundary conditions (1.6). At first, consideration will be limited to the case of half-spaces with identical parameters; for a system with different half-spaces closed results can be derived.

**2.** If the elastic properties of the half-spaces coincide, i.e. $a_0 = a_2$, $b_0 = b_2$, $\rho_0 = \rho_2$, then for the integrands in (1.7) the following expressions can be written down.

$$X_0 = \frac{1}{2}\left(\frac{L_3}{L_1} + \frac{L_4}{L_2}\right) e^{-kH\alpha_0} X_0^\circ, \qquad X_2^- = -\alpha_0 g\eta^2 \left(\frac{1}{L_1} + \frac{1}{L_2}\right) e^{-kH\alpha_0} X_0^\circ$$

$$Y_0 = \frac{1}{2}\left(\frac{L_5}{L_1} + \frac{L_6}{L_2}\right) e^{-kH\alpha_0} X_0^\circ, \quad X_1^- = \alpha_0 g\eta^2 \left(\frac{1}{L_1}\tanh\frac{kh\alpha_1}{2} + \frac{1}{L_2}\coth\frac{kh\alpha_1}{2}\right) e^{-kH\alpha_0} X_0^\circ$$

$$X_2 = \frac{1}{2}\left(\frac{L_7}{L_1} + \frac{L_8}{L_2}\right) e^{-kH\alpha_0} X_0^\circ, \qquad Y_2 = \frac{1}{2}\left(\frac{L_9}{L_1} + \frac{L_{10}}{L_2}\right) e^{-kH\alpha_0} X_0^\circ$$

$$L_1 = \alpha_1 R_0 + p_{10}\alpha_0\eta^4 \tanh\frac{kh\alpha_1}{2}, \qquad L_2 = \alpha_1 R_0 + p_{10}\alpha_0\eta^4 \coth\frac{kh\alpha_1}{2}$$

$$L_3 = p_{10}\alpha_0\eta^2 \tanh\frac{kh\alpha_1}{2} - \alpha_1 T_0, \qquad L_4 = p_{10}\alpha_0\eta^4 \coth\frac{kh\alpha_1}{2} - \alpha_1 T_0$$

$$L_5 = 4\alpha_0\alpha_1 g, \quad L_6 = -4\alpha_0\alpha_1 g, \quad L_7 = -L_3, \quad L_8 = L_3, \quad L_9 = L_5, \quad L_{10} = -L_5$$

$$R_0 = (2+\eta^2)^2 - 4\alpha_0\beta_0, \qquad T = (2+\eta^2)^2 + 4\alpha_0\beta_0, \qquad g = 2+\eta^2$$

$$p_{10} = \frac{\rho_1}{\rho_0}, \qquad \gamma_1 = \frac{a_1}{b_0}, \qquad \gamma_0 = \frac{a_0}{b_0} \qquad (2.1)$$

We note that, if in the formulas (2.1) the terms containing $L_1$ are discarded, a part of the wave field is obtained which is symmetric with respect to the plane $z = 1/2\ h$, and, similarly, if the terms containing $L_2$ are discarded the antisymmetric part remains.

These can be used to advantage for the investigation of the solution. The components of the displacement vectors in each of the media can be obtained from (1.7), if use is made of the equations

$$q_\nu = \frac{\partial\varphi_\nu}{\partial r} - \frac{\partial\psi_\nu}{\partial z}, \qquad w_\nu = \frac{\partial\varphi_\nu}{\partial z} + \frac{\partial\psi_\nu}{\partial r} + \frac{\psi_\nu}{r} \tag{2.2}$$

From equations (1.7), (2.1) and (2.2), the vertical components $w_1$ and the horizontal components $q_1$ of the displacement vector in the liquid layer can be represented by the following sum of the symmetric ($q_s$, $w_s$) and antisymmetric ($q_a$, $w_a$) parts of the wave field

$$q_1 \equiv q_s + q_a = \int_0^\infty \frac{kI_1(kr)\,dk}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} Q_{1s} X_0^\circ \exp\left(k\frac{t}{b_0}\eta - kH\alpha_0\right) d\eta + \int_0^\infty \frac{kI_1(kr)\,dk}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} Q_{1a} X_0^\circ \exp\left(k\frac{t}{b_0}\eta - kH\alpha_0\right) d\eta \tag{2.3}$$

$$w_1 \equiv w_s + w_a = \int_0^\infty \frac{kI_0(kr)\,dk}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} W_{1s} X_0^\circ \exp\left(k\frac{t}{b_0}\eta - kH\alpha_0\right) d\eta + \int_0^\infty \frac{kI_0(kr)\,dk}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} W_{1a} X_0^\circ \exp\left(k\frac{t}{b_0}\eta - kH\alpha_0\right) d\eta \tag{2.4}$$

$$Q_{1s} = -\frac{\alpha_0 g\eta^2}{L_2 \sinh kh\varkappa_1/2}\cosh\left[k\varkappa_1\left(z - \frac{h}{2}\right)\right], \qquad Q_{1a} = \frac{\alpha_0 g\eta^2}{L_1 \cosh kh\,\varkappa_1/2}\sinh\left[k\varkappa_1\left(z - \frac{h}{2}\right)\right]$$

$$W_{1s} = \frac{\alpha_0\varkappa_1 g\eta^2}{L_2 \sinh kh\varkappa_1/2}\sinh\left[k\varkappa_1\left(z - \frac{h}{2}\right)\right], \qquad W_{1a} = -\frac{\alpha_0 g\eta^2}{L_1 \cosh kh\varkappa_1/2}\cosh\left[k\varkappa_1\left(z - \frac{h}{2}\right)\right] \tag{2.5}$$

For the study of the solutions (2.3), (2.4), it is necessary to know the singularities of the integrands in (2.5) and, above all, the roots of the dispersional equations

$$L_1(kh, \eta) = 0, \qquad L_2(kh, \eta) = 0 \tag{2.6}$$

**3.** The equation $L_1(kh, \eta) = 0$, corresponding to antisymmetric vibrations, has exactly the same form as the dispersional equation indicated in [3] for the problem of the vibration of a liquid layer lying on an elastic half-space. There, it was proved that the low-frequency part of these vibrations can be represented by the low-frequency part of the interference Rayleigh wave, the phase velocities $v$ of which satisfy the inequality $c_2 > v > c_0$ (where $c_2$ is the velocity of the Rayleigh wave on the free surface of the elastic half-space, $c_0$ is the velocity of this same wave on the boundary between an elastic and a liquid half-space).

Thus, in such a system there are no low-frequency waves with a small velocity of propagation, which is the object of our discussion. Now we will consider the dispersion equation of the symmetric vibrations $L_2(kh, \eta) = 0$.

It is expedient to divide these roots into two classes, and into the first class are put the roots which are at a finite distance from the origin when $kh = 0$. It can be proved that, for an investigation of the low-frequency vibrations of stratified systems, it is sufficient to limit consideration to only the roots of the first class [4]. Multiplying the left-hand side of $L_2(kh, \eta) = 0$ by $kh$ and passing to the limit $kh \to 0$, it is not difficult to convince oneself that, when $kh = 0$, the equation has a root of fourth multiplicity at the origin and simple roots at the points $\pm i\gamma_0^{-1}$. As $kh$ increases, the root at the origin separates into two fixed roots and two simple roots $\eta = \pm iy(y > 0)$ moving out on the imaginary axis. An approximate expression for them when $kh$ is sufficiently small is obtained by expanding the function $L_2(kh, \eta)$ in a series of powers of $kh$ and $\eta$ and limiting the expansion to a few of the first terms

$$y = \left\{\frac{4(1-\gamma_0^2)\,kh}{4p_{10} + kh\,[2 + (1-\gamma_0^2)(1-\gamma_0^2+4\gamma_1^2)]}\right\}^{1/2} \tag{3.1}$$

As $kh$ increases, the ordinates of the roots increase monotonically and as $kh \to \infty$ they tend to the roots of the equation

$$\alpha_1(g_0^3 - 4\alpha_0\beta_0) + p_{10}\alpha_0\eta^4 = 0 \tag{3.2}$$

Equation (3.2) is obtained as the solution of the problem of the vibrations of the boundary separating a solid and liquid half-space. It has two roots on the imaginary axis with the ordinates $\pm\,\tau_0$ satisfying the inequality $\tau_0 < \vartheta$, where $\vartheta$ is the Rayleigh root for the free surface of an elastic half-space [5].

As can be seen from (1.7), the ordinates of the roots determine the phase velocity of the vibrations up to a constant multiplicative factor. Therefore, the phase velocities $v$ of the vibrations, corresponding to the roots considered, lie in the frequency range $0 < v < \tau_0/b_0$, which is of great interest. In regard to the roots moving out from $\pm\,i\gamma_0^{-1}$, they either move over to the second sheet of the Riemann surface (if $b_0^{-1} < a_0^{-1} < a_1^{-1}$), or they proceed to the left of the imaginary axis in the strip $\gamma_1^{-1} < |\,\mathrm{Im}\,\eta\,| < \gamma_0^{-1}$ (if the condition $b_0^{-1} < a_1^{-1} < a_0^{-1}$ or

$a_1^{-1} < b_0^{-1} < a_0^{-1}$ is satisfied), which leads to strong attenuation of the vibrations corresponding to these roots.

**4.** For the investigation of the integrals (2.3), (2.4) we will deform the path of integration of the inner integral into the left-hand half-plane. For this deformation we will have to cut through the branch line passing from the branch points and the pole of the integrands. Corresponding to this field of displacements, it will be possible to represent it as a sum of terms each of which describes a wave of a different physical nature (principal, reflected, Rayleigh, and such, waves). Now we will investigate the symmetric field which is described by the integral after substitution of the roots of the equation $L_2(kh, \eta) = 0$.

We will assume that the inequality $kh \ll 1$, equivalent to

$$\omega \ll (2a_1h)^{-1} \tag{4.1}$$

is satisfied. Then the function $L_2(kh, \eta)$ can be approximately represented in the following form

$$L_2 = \frac{2\eta^2 p_{01}}{kh}\left[\eta^2 + \frac{1-\gamma_0^2}{p_{01}}kh\right] \tag{4.2}$$

From (4.2) can be traced the anomalous character of the dispersion of the waves under consideration. The dependence of the phase velocity $v$ and the group velocity $u$ on the frequency are, by virtue of (4.1), determined by the formula

$$v = \left(\frac{1-\gamma_0^2}{p_{10}b_0^2}\right)^{\frac{1}{2}}(kh)^{\frac{1}{2}} = \left(\frac{1-\gamma_0^2}{p_{10}b_0^2}\right)^{\frac{1}{3}}(\omega h)^{\frac{1}{3}}, \qquad u = \frac{3}{2}v \tag{4.3}$$

It is interesting to recall that for the bending vibrations of a free layer and a layer in a liquid having the same anomalous dispersion, the phase velocities are proportional to $(\omega h)^{1/2}$ and $(\omega h)^{3/5}$, respectively.

The components of the displacement vector for symmetric vibrations, by (2.3), (2.4), satisfy the equalities

$$q_s = \frac{2}{\pi\sqrt{p_{01}(1-\gamma_0^2)}}\,\mathrm{Re}\int_0^{k_0}\frac{I_1(kr)\,k}{(kh)^{1/2}}e^{ikvt-kH}X_0^\circ\,dk \tag{4.4}$$

$$w_s = -\frac{2}{\pi\sqrt{p_{01}(1-\gamma_0)^2}}\,\mathrm{Re}\int_0^{k_0}\frac{I_0(kr)\,k^2}{(kh)^{1/2}}e^{ikvt-kH}\left(z-\frac{h}{2}\right)X_0^\circ\,dk \tag{4.5}$$

in which $v$ is determined by (4.3).

If the elastic properties of the half-spaces are different, the formulas for the displacement vectors and the phase velocities become somewhat complicated and can be represented in the form

$$v = (2\sqrt{1-\gamma_0^2 c})^{\frac{1}{3}} \left(\frac{\omega h}{b_0^2}\right)^{\frac{1}{3}}, \qquad q_1 = \frac{2\sqrt{2}}{\pi\sqrt{1-\gamma_0^2}} C^{\frac{1}{2}} \operatorname{Re} \int_0^{k_0} \frac{J_1(kr)\,k}{(kh)^{1/2}} e^{ikvt-kH} X_0^\circ dk \quad (4.6)$$

$$w_1 = -\frac{2\sqrt{2}}{\pi\sqrt{1-\gamma_0^2}} c^{\frac{1}{2}} \operatorname{Re} \int_0^{k_0} \frac{J_0(kr)\,k^2}{(kh)^{1/2}} e^{ikvt-kH} (z - \cosh) X_0^\circ dk \quad (4.7)$$

$$\left(c = \frac{\delta_2^2 - \gamma_2^2}{\rho_{10}(\delta_2^2 - \gamma_2^2) + \rho_{12}\delta_2^2(1-\gamma_0^2)}\right) \quad (4.8)$$

From (4.7) it follows that in the layer of liquid there is a plane $z = \cosh$, all the points of which do not undergo a vertical displacement during the propagation of the vibrations under consideration. Generally speaking, throughout the whole thickness of the layer the amplitudes $w_1$ of the components of the displacement field are smaller than the horizontal components by an amount $(z - \cosh)$, i.e. it could be said that the displacement vector has been polarized into one plane.

In the discussion of the properties of the part of the field written down, great importance is attached to the way in which the waves are attenuated with distance, We assume that the point of observation is situated at a sufficient distance, in order to have satisfied the inequality $kr \gg 1$, and we replace the Bessel function in (4.4) to (4.7) by the first term of the asymptotic expansion.

The result of integration can be written in the form

$$\frac{1}{r}\int_0^{K_0} \varphi(k) \sin\left\{r\left(k\frac{v}{V} \pm k\right) + \chi\right\} dk \quad (4.9)$$

where $\varphi(k)$ is slowly varying in comparison with the second factor. It is easy to prove that the displacement field (4.9) decreases

$$r^{-\frac{3}{2}} \text{ when } V > c_2, \qquad r^{-1} \text{ when } V < c_2$$

whereby $V$ is the translational velocity of the observer.

Thus, the wave disturbance is characterized by an unusually small attenuation with distance. The amplitude of the low-frequency part of the field does not depend on the velocity of propagation in the layer and is determined by the discontinuous jump in the densities of the media and the relation between the longitudinal and transverse velocities in the elastic medium. The amplitudes of the waves decreases exponentially with distance, when the ordinate of the point of observation of the source is away from the boundary of the layer, thus indicating the wave character of the propagation (this follows directly from (2.1)). The vibrations have a very low velocity of propagation, but because of the anomalous dispersion the higher frequencies will be propagated with a higher velocity.

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